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Asymptotic expression of the linear discrete best ℓ_p-approximation☆

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Abstract

Let h_p, $1 < p < \infty$, be the best ℓ_p-approximation of the element $h \in \mathbb{R}^n$ from a proper affine subspace K of $\mathbb{R}^n$, $h \notin K$, and let h^*_∞ denote the strict uniform approximation of h from K. We prove that there are a vector $\alpha \in \mathbb{R}^n \setminus \{0\}$ and a real number a, $0 \leqslant a \leqslant 1$, such that

$$h_p = h^*_\infty + \frac{a^p}{p-1}\alpha + \gamma_p,$$

for all $p > 1$, where $\gamma_p \in \mathbb{R}^n$ with $\|\gamma_p\| = o\left(a^p/p\right)$.

Keywords: Strict best approximation; Rate of convergence; Polya algorithm; Asymptotic expansion

1. Introduction

For $1 \leqslant p \leqslant \infty$, we consider the linear space $\mathbb{R}^n$ endowed with the usual p-norm. For convenience we will denote $\|\cdot\| := \|\cdot\|_\infty$. Also we will use the functional notation $x = (x(1), x(2), \ldots, x(n))$ to denote the element $x \in \mathbb{R}^n$.

Let $K \neq \emptyset$ be a subset of $\mathbb{R}^n$. For $h \in \mathbb{R}^n \setminus K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best ℓ_p-approximation of h from K if

$$\|h_p - h\|_p \leqslant \|f - h\|_p \quad \text{for all } f \in K.$$

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Throughout this paper, K will denote a proper affine subspace of $\mathbb{R}^n$ and, without loss of generality, we will assume that $h = 0$ and $0 \notin K$. In this context, the existence of h_p is guaranteed. Moreover, there exists a unique best ℓ_p-approximation if $1 < p < \infty$. In the case $p = \infty$ we will say that h_∞ is a best uniform approximation of 0 from K. In general, the unicity of the best uniform approximation is not guaranteed. However, a unique "strict uniform approximation", h^*_∞, can be defined [3,8]. For convenience we will write $K = h^*_\infty + \mathcal{V}$, where $\mathcal{V}$ is a proper linear subspace of $\mathbb{R}^n$. It is well known (see for instance [9]) that h_p, $1 < p < \infty$, is the best ℓ_p-approximation of 0 from K if and only if

$$\sum_{j=1}^{n} v(j) \left|h_p(j)\right|^{p-1} \operatorname{sgn}(h_p(j)) = 0 \quad \text{for all } v \in \mathcal{V}. \tag{1}$$

It is also known, [1,4,8], that $\lim_{p\to\infty} h_p = h^*_\infty$. This convergence is called Polya algorithm and occurs at a rate no worse than $1/p$ (see [2,4]). In [6] it is proved that for all $r \in \mathbb{N}$ there are $\alpha_l \in \mathcal{V}$, $1 \leqslant l \leqslant r$, such that

$$h_p = h^*_\infty + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)}, \tag{2}$$

where $\gamma_p^{(r)} \in \mathbb{R}^n$ and $\|\gamma_p^{(r)}\| = \mathcal{O}\left(p^{-r-1}\right)$. In [4] the authors give a necessary and sufficient condition on K for

$$p\,\|h_p - h^*_\infty\| \to 0 \quad \text{as } p \to \infty \tag{3}$$

and in [7] it is proved that if (3) holds then there are real numbers a, L_1 and L_2, with $0 \leqslant a < 1$ and $L_1, L_2 > 0$, such that

$$L_1 a^p \leqslant p\,\|h_p - h^*_\infty\| \leqslant L_2\, a^p \tag{4}$$

for all $p \geqslant 1$. In particular, (4) implies that if (3) holds, then we have an exponential rate of convergence of h_p to h^*_∞ as $p \to \infty$ and so the asymptotic expansion in (2) follows immediately with $\alpha_l = 0$, $1 \leqslant l \leqslant r$, for all $r \in \mathbb{N}$. The aim of this paper is to complete the results in [4,6,7] giving an asymptotic expression of h_p in the general case. More precisely, we prove that there does exist a vector $\alpha \in \mathcal{V}$, $\alpha \neq 0$, such that

$$h_p = h^*_\infty + \frac{a^p}{p-1}\,\alpha + \gamma_p, \tag{5}$$

where $\gamma_p \in \mathcal{V}$ and $\|\gamma_p\| = o\,(a^p/p)$.

In the case $0 < a < 1$, taking into account (4), we immediately deduce that $p\|h_p - h^*_\infty\|/a^p$ is bounded. However, it is not a trivial question to show that the limit $p(h_p - h^*_\infty)/a^p$ exists as $p \to \infty$. This justifies the present paper.

On the other hand, since there is trivially an expression of the form (5) for some α and γ_p in $\mathcal{V}$, the only part requiring proof is the error estimate for γ_p. Also observe that (5) is a particular case of (2) for $a = 1$. However, in the case $0 \leqslant a < 1$ expression (5) is specially interesting because (2) does not give any information about h_p.

2. Notation and preliminary results

Without loss of generality, we will assume that $\|h^*_\infty\| = 1$, $h^*_\infty(j) \geqslant 0$, $1 \leqslant j \leqslant n$, and that the coordinates of h^*_∞ are in decreasing ordering. Let $1 = d_1 > d_2 > \cdots > d_s \geqslant 0$ denote all the different values of $h^*_\infty(j)$, $1 \leqslant j \leqslant n$, and let $\{J_l\}_{l=1}^s$ be the partition of $J := \{1, 2, \dots, n\}$ defined by $J_l := \{j \in J : h^*_\infty(j) = d_l\}$, $1 \leqslant l \leqslant s$.

If $J' \subseteq J$ we will denote by $\|\cdot\|_{J'}$ the restriction of the norm $\|\cdot\|$ to the set of indices on J'.

Note that it is possible to choose a basis $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ of $\mathcal{V}$ and a partition $\{I_k\}_{k=1}^s$ of $I := \{1, 2, \dots, m\}$ such that for all $i \in I_k$, $1 \leqslant k \leqslant s$,

(p1) $v_i(j) = 0, \forall\, j \in J_l\,,\ 1 \leqslant l < k$,
(p2) $v_i(j) \neq 0$ for some $j \in J_k$.

Note that I_k can be empty for some k, $1 \leqslant k \leqslant s$. However, as we will notice later, the case $d_s > 0$ or $I_s = \emptyset$ simplify the proof of the results in this paper. For this reason, and to consider the more general situation, we will assume that $d_s = 0$ and $I_s \neq \emptyset$. We will use the following results.

Theorem 1 (*Quesada [7, Corollary 1]*). *Let*

$$a = \max_{1 \leqslant l,k \leqslant s-1} \left\{ d_l/d_k : \sum_{j \in J_l} v_i(j) \neq 0 \textit{ for some } i \in I_k \right\}, \tag{6}$$

where a is assumed to be 0 if $\sum_{j\in J_l} v_i(j) = 0$ for all $i \in I_k$, $1 \leqslant k, l \leqslant s-1$. Then there are $L_1, L_2 > 0$ such that

$$L_1\, a^p \leqslant p\,\|h_p - h^*_\infty\| \leqslant L_2\, a^p \quad \textit{for all } p \geqslant 1. \tag{7}$$

Lemma 2. *If $\{x_p\}$ is a sequence of real numbers such that $x_p \to 0$ as $p \to \infty$, then*

$$\left(1 + \frac{x_p}{p}\right)^p = 1 + x_p + R_p,$$

where $R_p = \mathcal{O}(x_p^2)$.

Proof. The proof follows immediately from the application of the Taylor's formula to the function $\varphi(z) = (1 + z/p)^p$ at $z = 0$. □

3. Asymptotic expression of the best ℓ_p-approximations

Since $h_p \to h^*_\infty$ as $p \to \infty$, then $h_p(j) > 0$ for all $j \in J_l$, $1 \leqslant l \leqslant s-1$, and p large enough. So, without loss of generality, we will assume that $h_p(j) > 0$ for all $j \in J_l$, $1 \leqslant j \leqslant s-1$.

Theorem 3. *Let K be a proper affine subspace of $\mathbb{R}^n$, $0 \notin K$. For $1 < p < \infty$, let h_p denote the best ℓ_p-approximation of 0 from K and let h^*_∞ be the strict uniform approximation. Let a be the real number defined in* (6). *Then there is a vector $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, such that*

$$h_p = h^*_\infty + \frac{a^p}{p-1}\,\alpha + \gamma_p, \tag{8}$$

where $\gamma_p \in \mathbb{R}^n$ and $\|\gamma_p\| = o(a^p/p)$.

Proof. Write $K = h^*_\infty + \mathcal{V}$, where $\mathcal{V}$ is a proper linear subspace of $\mathbb{R}^n$, and consider the basis $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ defined as above. By the conditions (p1) and (p2) and the definition of a in (6), we have $0 \leqslant a \leqslant 1$. We will consider three cases.

(a) If $a = 0$, then by (7), $h_p = h^*_\infty$ for all $p \geqslant 1$ and (8) follows trivially for all $\alpha \in \mathbb{R}^n$ and $\gamma_p = 0 \in \mathbb{R}^n$.

(b) If $a = 1$, then (8) is a particular case of (2), with $\gamma_p = \mathcal{O}(1/p^2)$. So, to conclude the proof, we only need to prove that $\alpha \neq 0$. Indeed, since $a = 1$, there exist $k \in \{1, 2, \ldots, s-1\}$ and $i \in I_k$ such that $\sum_{j\in J_k} v_i(j) \neq 0$. Applying (1) with $v = v_i$ we have

$$\sum_{j\in J} v_i(j)|h_p(j)|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0.$$

Since $v_i \in I_k$, taking into account (p1) and (8) the above equation can be written as

$$\sum_{j\in J_k} v_i(j)\left(d_k + \frac{\alpha(j)}{p-1} + \gamma_p(j)\right)^{p-1} + \sum_{l>k}^{s}\sum_{j\in J_l} v_i(j)\left|h_p(j)\right|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0.$$

Dividing by d_k^{p-1} and letting $p \to \infty$ we obtain $\sum_{j\in J_k} v_i(j)e^{\alpha(j)/d_k} = 0$ and hence $\alpha(j) \neq 0$ for some $j \in J_k$.

(c) If $0 < a < 1$, then $a = d_{l_0}/d_{k_0}$ for some $1 \leqslant k_0 < l_0 < s$ and $\sum_{j\in J_{l_0}} v_i(j) \neq 0$ for some $i \in I_{k_0}$. On the other hand, by Theorem 1, $p\,\|h_p - h^*_\infty\|/a^p$ is bounded. So, we can take a subsequence $p_k \to \infty$ such that $(p_k - 1)(h_{p_k} - h^*_\infty)/a^{p_k}$ converges. Define

$$\alpha := \lim_{k\to\infty} (p_k - 1)(h_{p_k} - h^*_\infty)/a^{p_k} \in \mathcal{V}.$$

By (7) $\alpha \neq 0$. Then we can write

$$h_p = h^*_\infty + \frac{a^p}{p-1}\alpha + \gamma_p, \tag{9}$$

where $\gamma_p := h_p - h^*_\infty - \alpha\, a^p/(p-1) \in \mathcal{V}$. Note that $p\,\|\gamma_p\|/a^p$ is also bounded and $p_k\,\|\gamma_{p_k}\|/a^{p_k} \to 0$ as $k \to \infty$. Now we prove that $\gamma_p = o\,(a^p/p)$. Indeed, suppose to the contrary that there exists a subsequence $p'_k \to \infty$ such that $(p'_k - 1)\gamma_{p'_k}/a^{p'_k} \to u \neq 0$. Since $u \in \mathcal{V}$, applying (1) with $v = u$ we have

$$\sum_{j\in J} u(j)\left|h_p(j)\right|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0. \tag{10}$$

Let $r_0 = \min\left\{l \in \{1, 2, \ldots, s\} : u(j) \neq 0 \text{ for some } j \in J_l\right\}$. Note that $u \in \mathrm{span}\{v_i : i \in I_k, r_0 \leqslant k \leqslant s\}$. Now we consider two cases:

(c.1) If $1 \leqslant r_0 \leqslant s-1$, then dividing (10) by $d_{r_0}^{p-1}$ and keeping in mind (9) we obtain

$$\begin{aligned}&\sum_{j\in J_{r_0}} u(j)\left(1 + \frac{\alpha(j)}{d_{r_0}}\frac{a^p}{p-1} + \frac{\gamma_p(j)}{d_{r_0}}\right)^{p-1}\\ &\quad + \sum_{l=r_0+1}^{s}\left(\frac{d_l}{d_{r_0}}\right)^{p-1}\sum_{j\in J_l} u(j)\left(1 + \frac{\alpha(j)}{d_l}\frac{a^p}{p-1} + \frac{\gamma_p(j)}{d_l}\right)^{p-1}\\ &\quad + \sum_{j\in J_s} u(j)\left|\frac{h_p(j)}{d_{r_0}}\right|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0.\end{aligned}$$

Now, applying Lemma 2 we get

$$\sum_{j\in J_{r_0}} u(j)\left(1+\frac{\alpha(j)}{d_{r_0}}\,a^p+(p-1)\frac{\gamma_p(j)}{d_{r_0}}+R_p(j)\right)$$
$$+\sum_{l=r_0+1}^{s-1}\left(\frac{d_l}{d_{r_0}}\right)^{p-1}\sum_{j\in J_l} u(j)\left(1+\frac{\alpha(j)}{d_l}\,a^p+(p-1)\frac{\gamma_p(j)}{d_l}+R_p(j)\right)$$
$$+\sum_{j\in J_s} u(j)\left|\frac{h_p(j)}{d_{r_0}}\right|^{p-1}\ \operatorname{sgn}(h_p(j))=0,$$

where $R_p(j)=\mathcal{O}\left(a^{2p}\right)$ for all $j\in J_l$, $r_0\leqslant l\leqslant s-1$.

If $r_0\leqslant l\leqslant s-1$ and $d_l/d_{r_0}>a$, then $d_l/d_r>a$ for all $r\geqslant r_0$. Hence, from the definition of a, $\sum_{j\in J_l} v_i(j)=0$ for all $i\in I_r$ with $r\geqslant r_0$ and hence $\sum_{j\in J_l} u(j)=0$. So, dividing by a^p and rearranging terms we can write the equality above as

$$\frac{1}{d_{r_0}}\sum_{j\in J_{r_0}} u(j)\alpha(j)+\frac{1}{a}\sum_{j\in J_{l_0}} u(j)+\frac{p-1}{a^p\,d_{r_0}}\sum_{j\in J_{r_0}} u(j)\gamma_p(j)+\widetilde{R}_p=0, \tag{11}$$

where $\widetilde{R}_p=\mathcal{O}\left(a^p\right)$ and l_0 is the possible index in $\{r_0+1,\ldots,s-1\}$ such that $d_{l_0}/d_{r_0}=a$.

Particularizing (11) for $p=p_k$ and taking limits as $k\to\infty$, we have

$$\frac{1}{d_{r_0}}\sum_{j\in J_{r_0}} u(j)\alpha(j)+\frac{1}{a}\sum_{j\in J_{l_0}} u(j)=0.$$

In similar way, letting $k\to\infty$ in (11) with $p=p'_k$ and taking into account the equality above, we obtain

$$\sum_{j\in J_{r_0}} u(j)^2=0.$$

A contradiction.

(c.2) If $r_0=s$, then multiplying (10) by $(p-1)^{p-1}/a^{p(p-1)}$ we get

$$\sum_{j\in\widehat{J_s}} u(j)\left|\alpha(j)+\frac{(p-1)\gamma_p(j)}{a^p}\right|^{p-1}\ \operatorname{sgn}\left(\alpha(j)+\frac{(p-1)\gamma_p(j)}{a^p}\right)$$
$$+\sum_{j\in J_s^0} u(j)\left|\frac{(p-1)\gamma_p(j)}{a^p}\right|^{p-1}\ \operatorname{sgn}(\gamma_p(j))=0, \tag{12}$$

where $\widehat{J_s}:=\{j\in J_s : u(j)\,\alpha(j)\neq 0\}$ and $J_s^0=J_s\setminus\widehat{J_s}$.

Note that $\widehat{J_s}\neq\emptyset$. Otherwise, particularizing (12) for $p=p'_k$ we get a contradiction for k large enough because $\operatorname{sgn}(\gamma_{p'_k}(j))=\operatorname{sgn}(u(j))$ if $u(j)\neq 0$.

Let $\widehat{J}_s^\alpha := \{j \in \widehat{J}_s : |\alpha(j)| = \|\alpha\|_{\widehat{J}_s}\}$. Dividing (12) for $\|\alpha\|_{\widehat{J}_s}^{p-1}$, we have

$$\sum_{j\in\widehat{J}_s^\alpha} u(j)\left|1+\frac{(p-1)\gamma_p(j)}{a^p\,\alpha(j)}\right|^{p-1} \operatorname{sgn}\left(\alpha(j)+\frac{(p-1)\gamma_p(j)}{a^p}\right)$$
$$+\sum_{j\in\widehat{J}_s\setminus\widehat{J}_s^\alpha} u(j)\left(\frac{\alpha(j)}{\|\alpha\|_{\widehat{J}_s}}\right)^{p-1}\left|1+\frac{(p-1)\gamma_p(j)}{\alpha(j)a^p}\right|^{p-1} \operatorname{sgn}\left(\alpha(j)+\frac{(p-1)\gamma_p(j)}{a^p}\right)$$
$$+\sum_{j\in J_s^0} u(j)\left|\frac{(p-1)\gamma_p(j)}{a^p\,\|\alpha\|_{\widehat{J}_s}}\right|^{p-1} \operatorname{sgn}(\gamma_p(j)) = 0. \tag{13}$$

Since $(p_k-1)\gamma_{p_k}(j)/a^{p_k} \to 0$ as $k\to\infty$, there exists a real number β, $0<\beta<1$, such that for k large enough

$$\min_{j\in\widehat{J}_s^\alpha}\left|1+\frac{(p_k-1)\gamma_{p_k}(j)}{a^{p_k}\alpha(j)}\right| > \beta > \max_{j\in\widehat{J}_s\setminus\widehat{J}_s^\alpha}\frac{\alpha(j)}{\|\alpha\|_{\widehat{J}_s}}\left|1+\frac{(p_k-1)\gamma_{p_k}(j)}{a^{p_k}\,\alpha(j)}\right|.$$

Now, taking into account that

$$-\sum_{j\in\widehat{J}_s^\alpha}\frac{|u(j)|}{\beta^{p_k-1}}\left|1+\frac{(p_k-1)\gamma_{p_k}(j)}{a^{p_k}\,\alpha(j)}\right|^{p_k-1} < -\sum_{j\in\widehat{J}_s^\alpha}|u(j)|$$

and

$$\lim_{k\to\infty}\frac{|u(j)|}{\beta^{p_k-1}}\left(\frac{\alpha(j)}{\|\alpha\|_{\widehat{J}_s}}\right)^{p_k-1}\left|1+\frac{(p_k-1)\gamma_{p_k}(j)}{\alpha(j)a^{p_k}}\right|^{p_k-1} = 0$$

for all $j\in\widehat{J}_s\setminus\widehat{J}_s^\alpha$, we deduce from (13) that there exists $j_0\in\widehat{J}_s^\alpha$ such that $u(j_0)\alpha(j_0)>0$. But, if $j\in\widehat{J}_s^\alpha$ and $u(j)\alpha(j)>0$, then

$$\lim_{k\to\infty}\left|1+\frac{(p_k'-1)\gamma_{p_k'}(j)}{a^{p_k'}\alpha(j)}\right| = 1+\frac{u(j)}{\alpha(j)} > 1$$

and $\operatorname{sgn}\left(\alpha(j)+(p_k'-1)\gamma_{p_k'}(j)/a^{p_k'}\right) = \operatorname{sgn}(u(j))$ for k large enough.

On the other hand, if $j\in\widehat{J}_s$ with $u(j)\alpha(j)<0$ and $|u(j)|\leqslant|\alpha(j)|$ then

$$\lim_{k\to\infty}\left|1+\frac{(p_k'-1)\gamma_{p_k'}(j)}{a^{p_k'}\alpha(j)}\right| = \left|1+\frac{u(j)}{\alpha(j)}\right| < 1.$$

Finally, if $j\in\widehat{J}_s$ with $u(j)\alpha(j)<0$ and $|u(j)|>|\alpha(j)|$ then

$$\operatorname{sgn}\left(\alpha(j)+(p_k'-1)\gamma_{p_k'}(j)/a^{p_k'}\right) = \operatorname{sgn}(u(j)).$$

So, taking limits in (13) as $k\to\infty$, with $p=p_k'$, we get a contradiction. □

Remark 4. Recently, in [5] it is proved that estimation (4) of the order of convergence of the Polya algorithm also holds if K is a finite affine subspace of $\ell_1(\mathbb{N})$. A slight modification of the techniques used in this paper shows that Theorem 3 is also valid in this new context.

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